A Study of Statistical Distributions using Fractional Calculus Approach

A
Synopsis

Submitted in

Partial fulfillment for the degree

of

Doctor of Philosophy

(MATHEMATICS)

Supervised by

Dr. Kishan Sharma

Submitted by

JV'n Ms. Nidhi Nagar

DEPARTMENT OF BIOTECHNOLOGY & ALLIED SCIENCES

FACULTY OF ENGINEERING AND TECHNOLOGY

JAYOTI VIDYAPEETH WOMEN'S UNIVERSITY,

JAIPUR (RAJASTHAN)

2015

INTRODUCTION

The present study is concerned with the essentials of fractional calculus that can be applied to the theory of probability and stochastic processes using varied approaches. The use of statistical distribution theory is all pervasive and has been successfully applied in a wide variety of contexts including the problems pertaining to various disciplines of physical, engineering, medical, biological and social sciences. The fractional calculus as a subject deals with the integration and differentiation of arbitrary real or complex orders. It can be understood as an extension of integer order calculus that finds applications in different areas of engineering and science including electric network, electrochemistry of corrosion, fluid flow, rheology, visco-elasticity to name a few. In statistical applications Poisson processes like the renewal processes, the distributions like power-law tail distribution, hyper geometric, gamma and exponential distributions, Cauchy distribution have been studied using a calculus approach.

The work in both of the above mentioned areas will not only yield a connection between the fractional calculus and statistical distribution theory but the rich results would also enrich the domains of fractional calculus for the development of the statistical distribution theory and vice versa. The proposed research work titled "A Study of Statistical Distributions using Fractional Calculus Approach" will attempt to study and explore means and ways to establish further connections and fusion of the two disciplines to open up a plethora of potential applications to find the solutions to the real world problems. The main purpose of the proposed research work is to study the use of Generalized Fractional Calculus (GFC) operators and functions of fractional calculus in solving various problems arising in science, engineering and technology and to determine its specific applications in the field of statistics. Further, we also wish to develop a lot of results in the area of fractional calculus and special functions that can be applied to statistical distribution theory through numerical computation.

The fractional calculus operators occur widely in technical problems associated with transmission lines, the theory of compression shocks waves etc. While the theory of fractional calculus operators is well developed, its uses have lagged behind particularly in the domain of statistics with a small number of researchers who have tried to associate the two streams together.

RESEARCH OBJECTIVES

The study aims to obtain new and useful results involving the special functions of fractional calculus by employing various fractional integral and differential operators as well as various integral transforms which have been defined recently. The solution of these equations will be investigated to yield a statistical interpretation. The study of various integral transforms such as Beta transform, Laplace transform, Mellin transform, Whittaker transform etcetera would be conducted and the efforts would be made to find out their possible new applications related to the statistical distribution theory through numerical computation. The specific objectives are to develop the statistical distribution theory in terms of special functions like Mittag-Leffler type functions.

REVIEW OF LITERATURE

Statistical Distributions & Fractional Calculus

The amalgam of the statistical distribution theory and fractional calculus owes heavily to the contributions of A.M. Mathai who established a connection between the two for augmenting the extant of applications of fractional calculus and statistical distribution theory and vice versa. His work establishing the statistical interpretations of fractional order integrals and derivatives along with the fractional integrals interpreted as the distributions of sums and differences of positive random variables as well as fractions of total probability opened up a number of opportunities to make use of the rich results in fractional calculus for further development of statistical distribution theory and vice versa. The subsequent sections introduce the various notations and definitions of various itineraries both from the statistical distributions' theory as well as fractional calculus that will be later employed in the study to gain a better understanding of the connection between the two domains. The aim of the present study is to develop a lot of results in the area of fractional calculus and special functions that can be applied to statistical distribution theory. The next section presents some basic concepts of statistical distribution theory that will be appearing in the proposed research work.

Statistical Preliminaries

Statistics is concerned with the collection of data and their analysis and their interpretations. After collecting the data we are interested to examine what the data tell us. Statistical inference means drawing conclusions based on data. More technically, parametric inference may be defined as the selection of an appropriate probabilistic model to describe the data that we wish to investigate, investigation of that model's behavior, and interpretation of the results. Two major approaches in inference are classical inference and Bayesian inference where both of them have their own significance.

Some works of Mathai are related to Statistical Mechanics in Physics. Nonequilibrium systems in various areas of science see Ebeling and Sokolov (2005) and Uffink (2007), exhibit spatio-temporal dynamics that is inhomogeneous and can be described by a superposition of several statistics on different scales, in short called a superstatistics approach in the physical literature, see Beck (2006). Essential for this superstatistics approach is the existence of sufficient time scale separation between two relevant dynamics within the nonequilibrium

system. There must be some intensive parameter that fluctuates on a larger time scale than the typical relaxation time of the local dynamics.

Generalized Gamma Density

When building models for the data in hand one usually takes a member from a parametric family of functions. It is often found that the data require a model with a thicker or thinner tail or of a slightly different shape in general. One way of controlling the tail behavior of a given gamma model is to append a Bessel series to it so that one can create models with thicker or thinner tails compared to the original gamma model without the appended Bessel series. The present literature review discusses the generalized gamma type models and their various applications in different disciplines.

The generalized gamma density is a continuous density with three parameters. It is a generalization of the two-parameter gamma density. The gamma density is the most popular model for analyzing skewed data. The generalized gamma density (GG), as introduced by Stacy (1962), includes as special sub-models: the exponential, Weibull, gamma and Rayleigh distributions, among others. The GG density is suitable for modeling data with different types of hazard rate functions: increasing, decreasing, in the form of bathtub and unimodal. This characteristic is useful for estimating individual hazard rate functions and both relative hazards and relative times (Cox, 2008). Recently, the GG density has been used in several research areas such as engineering, hydrology and survival analysis. Ali et al. (2008) derived the exact densities of the product $x_1 x_2$ and the quotient x_1/x_2, when x_1 and x_2 are independent GG random variables providing applications of their results to drought data from Nebraska. While not as frequently used for modeling life data as the previous densities, the generalized gamma density does have the ability to mimic the attributes of other densities such as the Weibull or lognormal, based on the values of the density's parameters. While the generalized gamma density is not often used to model life data by itself, its ability to behave like other more commonly used life models is sometimes used to determine which of those life distributions should be used to model a particular set of data.

Hwang and Huang (2006) introduced a new moment estimation of parameters of the generalized gamma density using its characterization. In information theory, thus far a maximum entropy (ME) derivation of GG is found in Kapur (1989), where it is referred to as generalized Weibull density, and the entropy of GG has appeared in the context of flexible families of

densities Nadarajah and Zografos (2003). Some concepts of this family in information theory have been introduced by Dadpay et al. (2007). Khodabin and Ahmadabadi (2010) studied GG entropy representation and its estimation. More recently, Barkauskas et al. (2009) modeled the noise part of a spectrum as an autoregressive moving average (ARMA) model with innovations having the GG density. Due to its long history and growing use in various applications, the GG family and its properties have been remarkably presented in different papers.

The density function of the generalized gamma (GG (a, t, λ)) is given by

$$f(y|a, t, \lambda) = t\,\lambda G(a)\,(y\lambda)\,at\text{-}1e\text{-}(y\lambda)t\ ,\ y = 0, a, t, \lambda > 0 \quad (2.2.1)$$

where a and t are shape parameters, and λ is the scale parameter. The GG family is flexible in that it includes several well-known models as subfamilies. The subfamilies of GG thus far considered in the literature are exponential (a = t = 1), gamma for (t = 1), and Weibull for (a = 1). The lognormal distribution is also obtained as a limiting distribution when a → 1. By letting t = 2 we obtain a subfamily of GG which is known as the generalized normal model (GN). The GN is itself a flexible family and includes half-normal (a = ½, $\lambda^2 = 2\sigma^2$), Rayleigh (a = 1, $\lambda^2 = 2\sigma^2$), Maxwell-Boltzmann (a =3/2) and chi (a =k/2, k = 1, 2, · ·). The subfamily of this density is introduced briefly as follows.

Exponential density

The exponential density occurs naturally when describing the lengths of the inter-arrival times (waiting times) in a homogeneous Poisson process. In queuing theory, the service times of agents in a system (e.g. how long it takes for a bank teller etc. to serve a customer) are often modeled as exponentially distributed variables. Reliability theory and reliability engineering also make extensive use of the exponential density. Because of the memory-less property of this density, it is well-suited to model the constant hazard rate portion of the bathtub curve used in reliability theory.

Gamma density

In probability theory and statistics, the gamma density is a two-parameter family of continuous density. It has a scale parameter λ and a shape parameter a. If a is an integer then the density represents the sum of a independent exponentially distributed random variables, each of which has a mean of λ (which is equivalent to a rate parameter of λ^{-1}). The gamma density is frequently a probability model for waiting times; for instance, in life testing, the waiting time until death is a random variable that is frequently modeled with a gamma density.

Weibull density

The Weibull density is a continuous probability density. It is named after Waloddi Weibull who described it in detail in 1951, although it was first identified by Frechet in 1927. The Weibull distribution is often used in the field of life data analysis due to its ability to fit the exponential density and the normal density and interpolate a range of shapes in between them. It is available from a power transformation of the exponential density.

Generalized Normal density

The GN family includes some well-known models as subfamilies.

Half normal density

The half-normal density is the density of the absolute value of a random variable that is normally distributed with expected value 0 and variance σ^2 i.e. if x is normally distributed with mean 0, then y = |x| is half-normally distributed.

Rayleigh density

In statistical literature, Rayleigh density is a continuous density. As an example of how it arises, the wind speed will have a Rayleigh density if the components of the two-dimensional wind velocity vector are uncorrelated and normally distributed with equal variance. The density is named after Lord Rayleigh.

Maxwell-Boltzmann density

The Maxwell-Boltzmann density applies to ideal gases close to thermodynamic equilibrium, negligible quantum effects, and non-relativistic speeds. It forms the basis of the kinetic theory of gases, which explains many fundamental gas properties, including pressure and diffusion. The Maxwell-Boltzmann density is usually thought of as the density for molecular speeds, but it can also refer to the density for velocities, momenta, and magnitude of the momenta of the molecules, etc. The original derivation by Maxwell assumed all three directions would behave in the same fashion, but a later derivation by Boltzmann dropped this assumption using kinetic theory. The Maxwell-Boltzmann density can now most readily be derived from the Boltzmann distribution for energies.

Chi-square density

It is Karl Pearson (1857-1936), the father of modern statistics, who came up with the Chi-square density. Pearson's work in statistics began with developing mathematical methods for studying the processes of heredity and evolution (leading to his aggressive advocacy of

eugenics). The Chi-square distribution came about as Pearson was attempting to find a measure of the goodness of fit of other distributions to random variables in his heredity and evolutionary modeling. In probability theory, the chi-square density ($\lambda = 1$ in the Chi density) with k degrees of freedom is the density of a sum of squares of k independently distributed standard normal random variables. It is one of the most widely used probability models in inferential statistics, e.g. in hypothesis testing, goodness-of-fit tests, independence of two criteria of classification of qualitative data, estimating variances, analysis of variance problems via its role in the F-distribution and so on. The sum of squares of statistically independently distributed Gaussian variables which do not have mean zero and unit-variance yields a generalization of the chi-square density called the non-central chi-square density.

Generalized Laplacian models

For the sake of completeness we the Laplacian model is also discussed. An analysis of the general input-output structure and generalized Laplacian model is given in Mathai (1993a,b). Let x_1 and x_2 be two independently distributed real gamma variables with parameters (α_1, β_1) and (α_2, β_2) respectively, $\alpha i>0$, $\beta i>0$, i= 1, 2. Let u =. Due to the independence of x_1 and x_2 the moment generating function of u is given by

$M_{\underline{u}}(t) = (1 - ß_{1t})\text{-}a_1 (1 + ß_{2t})\text{-}a_2, ß_2)$.

It is evident that a central or non-central generalized Laplacian model can arise from a difference of the type $u = x_1 - x_2$, where x_1 and x_2 are independently distributed gamma variables. Let x_1 represent the input and x_2 the output then $u = x_1 - x_2$ can be interpreted as the residual effect or gain or loss. Thus NGL can represent the exact distribution of this residual effect or gain or loss. This concept is connected to bilinear forms, quadratic forms and the concept of chi-squaredness of quadratic forms, which is the basis for making inference in analysis of variance, analysis of covariance, regression and general model building areas (Mathai, 1993a, b; Mathai et al., 1995).

Mathematical Preliminaries

A unified approach to the so-called fractional calculus and statistical distribution theory has received increasing interest off-late from both theoretical mathematicians and applied scientists. Special Functions (SFs) constitute a very old branch of mathematics; the origins of their unified and rather complete theory date to the nineteenth century. Most of the named functions provide solutions to integer order differential equations and systems, used as

mathematical models. However, recently there has been an increasing interest in and widely extended use of differential equations and systems of fractional order, as better models of phenomena of various physical, engineering, automatization, biological and biomedical, chemical, Earth science, economic etc. The subsequent section is allotted to special functions and fractional calculus that will find mention in the proposed research work.

Fractional Calculus

Fractional Calculus is a field of mathematic study that grow out of the traditional definitions of the calculus integral and derivatives operators in much the same way fractional exponents is an outgrowth of exponents with integer value. As one considers the physical meaning of the exponent as yielded by our primary school teachers, exponents are comprehended as a short notation for what is essentially a repeated multiplication of a numerical value. This concept in itself may be easy to grasp and is straight forward. However, this physical definition can clearly become confused when exponents are of non integer value.

A Fractional calculus has its origin in the question of the extension of meaning. A well known example is the extension of meaning of real numbers to complex numbers, and another is the extension of meaning of factorials of integers to factorials of complex numbers. In generalized integration and differentiation the question of the extension of meaning is: Can the meaning of derivatives of integral order $\frac{d^n y}{dx^n}$ be extended to have meaning where n is any number irrational, fractional or complex. Leibnitz invented the above notation. It was naive play with symbols that prompted L'Hospital to ask Leibnitz about the possibility that n be a fraction. "What if n be ½? ", asked L'Hospital. Leibnitz in 1695, replied, "It will lead to a paradox." But he added prophetically, "From this apparent paradox, one day useful consequences will be drawn." In 1697, Leibnitz, referring to Wallis's infinite product for $\pi/2$, used the notation $d^{1/2} y$ and stated that differential calculus might have been used to achieve the same result.

Fractional derivatives of multivariable generalized hyper-geometric function or the fractional derivatives (and fractional integral) of special function involving one and more variable are important such as in the evolution of series and integrals the derivation of generating function and the solution of the integral equation motivated by these and many other avenues of applications the fractional calculus operator ${}_{\alpha}D_x^{\mu}$ is much used in the theory of special function of one and more variables.

Fractional calculus is the field of mathematical analysis which deals with the investigation and application of integrals and derivatives of arbitrary order. It may be considered as an old but novel topic. It is an old topic since, starting from the speculations of Liebnitz (1695) and Euler (1738); it has been developed up to the current level. The mathematicians who have provided important contributions up to the middle of the 20th century are Laplace (1812), Fourier (1827), Abel (1823), Liouville (1832, 1837), Riemann (1876), Holmgren(1865), Grunwald (1867), Letnikov (1868), Laurent (1884), Nekrassov (1888), Krug (1890),Heaviside (1892), Pincherele (1888), Hardy and Littlewood (1926), Weyl (1917), Levy (1954), Marchaud (1927), Davis (1936), Zygmund (1945), Love (1967), Erdelyi (1939), Kober (1940), Riesz (1949), Feller (1952).

The first application of fractional calculus belongs to Niels Henrik Abel in 1823. Abel applied the fractional calculus in the solution of an integral equation which arises in the formulation of the tautochrone problem. Between 1835 and 1850 there was a controversy which centered on two definitions of a fractional derivative concerning the George Peacock Facord Lacroix's generalization of a case of integral order. Riemann in 1847 while a student wrote a paper published posthumously in which he gives a definition of a fractional operation. Cayley remarked in 1880 that Riemann's complementary function is of indeterminate nature.

Fractional calculus is the branch of mathematical analysis that studies the possibility of taking real number powers to complex number powers of integral and differential operators which we shall term as differintegral operators. Although the mathematical ideas of fractional calculus were developed long ago by the mathematicians Leibniz (1962), Liouville (1837), Riemann (1876), and others but the attention of the engineering world was brought by Oliver Heaviside in 1890's. In 1974, the first book on the topic was published by Oldham and Spanier. Recent monographs and symposia proceedings have highlighted the application of fractional calculus in physics, continuum mechanics; signal processing, electromagnetics and bioengineering.

The basic mathematical ideas of fractional calculus (integral and differential operations of non-integer order) were developed long ago by the mathematicians Leibniz (1695), Liouville (1834), Riemann (1892), and others and brought to the attention of the engineering world by Oliver Heaviside in the 1890s, it was not until 1974 that the first book on the topic was published by Oldham and Spanier. Recent monographs and symposia proceedings have highlighted the

application of fractional calculus in physics, continuum mechanics, signal processing, and electromagnetic. Here we state some of applications.

It may be important to point out that the first application of fractional calculus was made by Abel (1802-1829) in the solution of an integral equation that arises in the formulation of the *tautochronous problem*. This problem deals with the determination of the shape of a frictionless plane curve through the origin in a vertical plane along which a particle of mass can fall in a time that is independent of the starting position.

During the last decades of the nineteenth century, Heaviside success-fully developed his operational calculus without rigorous mathematical arguments. In 1892 he introduced the idea of fractional derivatives in his study of electric transmission lines. Based on the symbolic operator form solution of heat equation due to Gregory (1846), Heaviside introduced the letter p for the differential operator $\frac{d}{dt}$ and gave the solution of the diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = a^2 p$$

for the temperature distribution $u(x, t)$ in the symbolic form

$$u(x,t) = A\exp(ax\sqrt{p}) + B\exp(-ax\sqrt{p})$$

In which $p \equiv \frac{d}{dx}$ was treated as constant, where a, A and B are also constant.

Fractional calculus is used to describe the viscous interactions between fluid and solid structure. Reflection and transmission scattering operators are derived for a slab of cancellous bone in the elastic frame using Blot's theory. Experimental results are compared with theoretical pre-dictions for slow and fast waves transmitted through human cancellous bone samples

The methods of fractional calculus, when defined as a Laplace or Fourier convolution product, are suitable for solving many problems in biomedical research. For example, early studies of the electrical properties of nerve cell membranes and the propagation of electrical signals are well characterized by differential equations of fractional order. The solution involves the Mittag-Leffler function, which provides a better fit to the observed cell membrane data. A parallel application of fractional derivatives to viscoelastic materials establishes, in a natural way, hereditary integrals and the power law stress-strain relationship for modeling biomaterials. Many applications of fractional calculus in physics are amount to replacing the time derivative in

evolution equation with a derivative of fractional order. This is not merely a phenomenological procedure providing an additional fit parameter, rather fractional calculus seem to arise generally and universally and for deep mathematical reasons.

The fractional differentiation and fractional integration goes back to many mathematicians such as Leibniz, Liouville, Grunewald, Letnikov, Riemann, Abel, Riesz, and Weyl. The integrals and derivatives of non-integer order, and the fractional integro-differential equations have found many applications in recent studies in theoretical physics, mechanics and applied mathematics. There exists the remarkably comprehensive encyclopedic-type monograph by Samko, Kilbus and Marichev, which was published in Russian in 1987 and in English in 1993. The works devoted substantially to fractional deferential equations are the book by Miller and Ross (1993), and the book by Podlubny (1999). In 2006 Kilbas, Srivastava and Trujillo published a very important and remarkable book, where one can find a modern encyclopedic, detailed and rigorous theory of fractional deferential equations. The first book devoted exclusively to the fractional dynamics and application of fractional calculus to chaos is the book by Zaslavsky published in 2005.

Special Functions

The Special Functions of Fractional Calculus (SFs of FC) have drawn great scholastic attention off late equally from both the theoretical mathematicians as well as the applied scientists. Usually we call a function 'special' when the function belongs to the toolbox of the applied mathematician, the physicist or the engineer. They have a particular notation and a number of properties. Mathematically, special functions are functions defined on < or C and they possess not only series representations, but also integral representations.

This thesis is mainly concerned with the development of generalized Mittag-Leffler type functions such as generalized M-Series. The function plays significant role in statistical modeling. So the concept of Pochhammer notation, Mellin-Barnes integrals, convergence and residue calculus are essential for the detailed study of these functions. Recently the attention of mathematicians towards these functions has increased from both the analytical and numerical point of view due to their relation with the fractional calculus.

Thus the thesis also relies on various fractional differential equations such as fractional Poisson equation, fractional Laplace equation, fractional kinetic equation or the anomalous relaxation equation, the fractional Helmholtz equation etecetra. Now a days special functions

play an important role in the emerging field of graph theory, namely the graph dynamics in social networking and electronic banking (Flomenbom & Klafter,2005; Beghin & Orsingher, 2009; Roberto et al., 2011) .Charles's Fox in 1961 extensively studied a certain Mellin-Barnes integral, what was available in the literature from the nineteenth century (Pincherle, 1888; Mellin, 1910; Barnes, 1908), and hence this Mellin-Barnes integral is now known as Fox's H-function or simply the H-function. This function includes among others the recently found special functions by Boersma (1962), Wright (1934), Kratzel (1979), Mittag-Leffler function and its generalizations by Dzherbashyan (1960), Kiryakova (2000) and Prabhakar (1971), and the Meijer G-function (1946). The works of Caputo (1969), Glockle and Nonnenmacher (1993), Mainardi et al. (2001), Metzler and Klafter (2000), Podlubny (1999), Saichev and Zaslavasky (1997), Mathai et al. (2010), Scheider and Wyss (1989) show that it has a wider, deeper and useful application in various problems in applied sciences such as fluid flow, rheology, diffusion in porous media, kinematics in viscoelastic media, relaxation and similar other areas.

The theory of special functions has a long and varied history with immense literature due to their applicability as solutions of arbitrary order and the fractional order (or multi-order) differential and integral equations modeling the various phenomena arising in physical, biological, chemical, engineering and earth sciences along with problems pertaining to automatization and economics mathematically.

The Special Functions (with acronym SFs) comprise a very old branch of mathematics. The origin of their completely unified theory dates back to the nineteenth century. Special functions have an origin in the solution of partial differential equations satisfying certain prescribed conditions. At present special functions are defined in several ways notably by power series, generating functions, infinite products, integrals, difference equations, trigonometric or orthogonal function series namely the Airy functions; the Bessel and all cylindrical functions; the Gauss, Kummer, confluent and generalized hypergeometric functions; the classical orthogonal polynomials, the incomplete Gamma- and Beta-functions and Error functions, Whittaker functions and so on. From the perspective of the applied scientists who deal with the practical applications of differential equations, SFs serve as an important tool of mathematical analysis as they yield solutions to integer order(!) differential equations and systems and are used for mathematical modeling.

Eminent mathematicians notably Euler, Legendre, Gauss, Jacobi, Weierstrass, Kummer, Riemann, Ramanujan worked hard to develop special functions like Bessel functions, Whittaker functions, Gauss hyper geometric function and the polynomials that go by the names of Jacobi, Legendre, Laguerre, Hermite etc. The Gaussian hyper geometric function ${}_pF_q$ and its special cases are commonly used in applied mathematics and mathematical physics. Since ${}_pF_q$ diverges for $p > q+1$, in an attempt to give meaning to it in this case , MacRobrt (1959) and Meijer (1946) introduced the E-function and the G-function, respectively. Though these functions are quite general in character, a number of special functions like Wright's generalized hyper geometric function, generalized parabolic cylinder function, Mittag-Leffler function, and several other functions, do not form their special cases.

The famous handbooks of ``Classical Calculus" era constitutes the definitions, properties and well known examples of the Special Functions of Mathematical Physics (which are also referred as ``Classical Special Functions"), one important publication in this regard is the Bateman Project. The contemporary perspective of these SFs can be summarized in the form of hyper geometric ${}_pF_q$-functions or Meijer's G-functions. The presentation regarding these extended lists of representations is published in the handbooks and monographs after 1980. The importance of these G-functions is inherent in their tables as few of their fundamental properties enable a beginner to explore the functional relationships between two special functions, evaluation of their differential, integrals and asymptotic properties, to determine the solutions of integer order differential equations from the general ODE that satisfy the Meijer G-functions regardless of the unique facts on these named functions. The contemporary software packages designed for analytical purposes (like Matlab, Maple, and Mathematica) allow the recognition as well as working of the classical SFs in an efficient manner. The alternative perspective on the Classical SFs makes use of the Fractional Calculus operators for reducing them to the well known and simpler special/basic elementary functions for example the one like the Kiryakova's Generalized Fractional Calculus (GFC).The Special Functions of Fractional Calculus are in dire need of unified presentation and extensive development. The following section discusses some important definitions.

Mittag-Leffler Functions

The Mittag-Leffler function was defined and studied by the Swedish mathematician G. M. Mittag-Leffler in 1903. It is a direct generalization of the exponential series. Later Wiman

(1905), Agarwal (1953) and Humbert and Agarwal (1953) generalized and studied the properties of this function. Dzherbashyan (1966) has shown that both the functions are entire in C of order $\rho = 1/\alpha$ and type $\sigma = 1$. The basic properties of these functions are given in Erdelyi et al. (1955). A generalized form of Mittag-Leffler function was introduced by Prabhakar (1971). Firstly in 1920 Hille and Tamarkin have presented a solution of Abel-Volterra type integral equation in terms of Mittag-Leffler function.

Purohit, Kalla and Suthar (2011) derived some properties of multiindex Mittag-Leffler function introduced by Kriyakova and they also established certain theorems which provide the image of this function under the Saigo's fractional integral operators. The results derived one of the general character and give rise to a number of known results in the theory of multiindex Mittag-Leffler functions. Shukla and Prajapati (2009a) investigated a recurrence relation and integral representation of generalized Mittag-Leffler function. Saxena, Ram and Vishnoi (2011) studied generalized Mittag-Leffler function introduced by Saxena and Nishimota (2010) integral representations of generalized Mittag-Leffler function are derived. The result obtained provides new result for the Mittag-Leffler functions studies by Shrivastva and Tomovski (2009) and Shukla and Prajapati (2007; 2009). Chaurasia and Kumar (2010) established an integral pertaining to a product of Fox H-function (Fox, 1961), M-series (Sharma, 2008), general polynomials (Srivastava & Panda, 1996) with general arguments of quadratic nature. Sharma (2008) introduced the M-series and investigated the fractional integration and differentiation of that series. Sharma and Jain (2009) introduced the generalized M-series and investigated the fractional integration and differentiation of that series. Sharma and Dhakar (2011) introduced a new K_2- function and derived the relations that exist between the K_2-function and the operators of Riemann- Liouville fractional integrals and derivatives. Sharma (2011) derived the solution of generalized fractional kinetic equation. Results obtain in a compact form in terms of K_4-function. Sharma (2012) introduced a new function namely K_4-function and demonstrated how K_4-function is closely related to another special functions. The differintegration of that function is also investigated. Sharma (2012) investigated the generalized fractional integration of the generalized M-series and some results derived by Saxena and Siogo (2005), Samko, Kilbas and Marichev (1993) are the special case of the main results derived in this paper. Sharma, Jain and Sharma (2011) studied the derivation of certain transformation formulae for the basic

hypergeometric functions as an application of fractional q-derivative. Sharma (2011) derived a set of generalized fractional Volterra type integral equation involving K_4-function.

Applications of Mittag-Leffler Type functions

Mittag-Leffler distributions can be used as waiting-time distributions as well as first-passage time distributions for certain renewal processes with geometric exponential as waiting-time distribution. Mittag-Leffler functions are also used for computation of the change of the chemical composition in stars like the Sun. Recent investigations have proved that Mittag-Leffler functions are certain aspects of the solution of the solar neutrino problems in astrophysics, which can be expressed in terms of special functions like G and H-functions (Saxena et al., 2004a,b; Sebastian, 2011). The Mittag-Leffler distribution finds applications in wide range of contexts such as stress-strength analysis, growth-decay mechanisms like formation of sand dunes in nature, input-output situations in economics, crop prices, industrial productions, production of melatonin in human body etc. In recent years the fractional generalization of the classical Poisson process has gained increasing interest. In it the waiting time between events is given by the Mittag-Leffler distribution function F in place of the exponential distribution. Of all the papers devoted to this special renewal process we content ourselves to cite only Mainardi et al. (2004) and Gorenflo and Mainardi (2012). In Saxena et al. (2004) it is shown that the fractional generalization of the linear kinetic-type equation also leads to power-law behavior and the solutions can be expressed in terms of generalized Mittag-Leffler functions.

METHODOLOGY

In the proposed work, the research work would be emphasizing upon the extensive literature survey using the departmental libraries, journals, books, academic references and other secondary data sources. The study would commence with the in-depth analysis of the previously existing work in the aforementioned context and gaining an exhaustive understanding of the results that have been derived in the past. For the purpose the synthesis of the recent studies on the aforesaid areas would be done by examining the academic articles that have been published in the period 2005-2015 using electronic database. The findings drawn from the review of literature would be presented in the Chapter 1 of the thesis. Then expert surveys with the researchers, academicians and subject scholars would be done to procure the inputs from the various experts on the subject to strengthen the comprehension of the matter summarized as well to gain valuable insights about developing it further according to the proposed research objectives. Last but not the least new results on the proposed objectives would be derived using numerical computations.

REFERENCES

1. Agrawal, P. and Koul, C. L., Relation Involving PSI Function and H-Function of Several Variables Journal of Indian Acad. Maths. Vol.28 (2006), 63-72.
2. Babenko, Yu. I., Heat and Mass Transfer, Chimia, Leningrad, (1986).
3. Barnes, E. W., The asymptotic expansion of integral functions defined by Taylor series. Phil. Trans. Roy. Soc. London A. 206, (1908), 249-297.
4. Bassam, M.A. Al., Luchko, Yu. F., On Generalized Fractional Calculus and its Application to the Solution of Integro-differential Equation, Journal of Fractional Calculus Vol. 7 (1995), 69-88.
5. Beghin, L. and Orsingher, E., Fractional Poisson processes and related planar random motions, Electronic Journal of Probability. 14, (2009) 17901826.
6. Chakraborty, Subrata, and S. H. Ong. "Mittag-Leffler function distribution-A new generalization of hyper Poisson distribution." *arXiv preprint arXiv: 1411.0980*(2014).
7. Chaurasia and Kumar, The Integration of Certain Product Involving Special Functions, Scientia Series A: Mathematical Science, Vol.19 (2010), 7-12.
8. Das, S., Functional Fractional Calculus for System Identification and Controls, Springer, Berlin,Heidelberg, New York (2008).
9. Erdélyi, A., Tables of Integral Transforms Vol. II McGraw Hill, New York, Toronto and London (1954).
10. Exton, H., A New Hypergeometric Function, Journal of the Indian Acad. Math. Vol. 21 (1999), 53. -58.
11. Fox, C., The G and H-Function as Symmetrical Fourier Kernel, Trans. Amer. Math.Soc., 98 (1961), 395-429.
12. Gorenflo, Rudolf, Anatoly A. Kilbas, Francesco Mainardi, and Sergei V. Rogosin. "Mittag-Leffler Functions with Three Parameters." In *Mittag-Leffler Functions, Related Topics and Applications*, Springer Berlin Heidelberg, (2014), 97-128.
13. Gorenflo, Rudolf, Anatoly A. Kilbas, Francesco Mainardi, and Sergei V. Rogosin. "Historical Overview of the Mittag-Leffler Functions." In *Mittag-Leffler Functions, Related Topics and Applications*, Springer Berlin Heidelberg, (2014), 7-16. Gorenflo, Rudolf, Anatoly A. Kilbas, Francesco Mainardi, and Sergei V. Rogosin. "Applications to

Deterministic Models." In *Mittag-Leffler Functions, Related Topics and Applications*, Springer Berlin Heidelberg, (2014), 201-233.

14. Gorenflo, Rudolf, Anatoly A. Kilbas, Francesco Mainardi, and Sergei V. Rogosin. "Applications to Stochastic Models." In *Mittag-Leffler Functions, Related Topics and Applications*, Springer Berlin Heidelberg, (2014), 235-268.
15. Hartley, Tom T. and Lorenzo, Carl F., A Solution to the Fundamental Linear Fractional Order Differential Equations, NASA/TP-1998-208693 (1998), 16p; Available electronically at http;//gltrs.grc.nasa.gov/reports/1998/TP-1998-208693.pdf
16. Haubold, Hans J., Arak M. Mathai, and Ram K. Saxena. "Mittag-Leffler functions and their applications." *Journal of Applied Mathematics* 2011 (2011).
17. Hussain, M. M., and Belgacem, F. M., Transient Solutions of Maxwell's Equations Based on Sumudu Transform, Progress In Electromagnetics Research, PIER 74(2007), 273–289.
18. Kilbas, A. Anatolii Aleksandrovich, Hari Mohan Srivastava, and Juan J. Trujillo.*Theory and applications of fractional differential equations*. Vol. 204. Elsevier Science Limited, 2006.
19. Kilbas, Anatoly A. *H-transforms: Theory and Applications*. CRC Press, 2004.
20. Lorenzo, Carl F. and Hartley, Tom T., Generalized Functions for the Fractional Calculus, NASA/TP-1999-209424/REV1,(1999), 17p.;Available electronically at http://gltrs.grc.nasa.gov/reports/1999/TP-1999-209424.pdf
21. Mathai, A. "Some properties of Mittag-Leffler functions and matrix-variate analogues: A statistical perspective." *Fractional Calculus and Applied Analysis*13, no. 2 (2010): 113-132.
22. Mathai, A. M. "A pathway to matrix-variate gamma and normal densities."*Linear Algebra and Its Applications* 396 (2005): 317-328.
23. Mathai, Arakaparampil M., and Hans J. Haubold. Special functions for applied scientists. Springer Science+ Business Media, 2008.
24. Miller, K.S. and Ross B., An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, New York etc.(1993).
25. Oldham, K .B. and Spanier, J. The Fractional Calculus, Academic Press, New York and London (1974)
26. Özarslan, Mehmet Ali, and Banu Yılmaz. "The extended Mittag-Leffler function and its properties." *Journal of Inequalities and Applications* 2014, no. 1 (2014): 1-10.

27. Prabhakar, T.R., A Singular Integral Equation with a Generalized Mittag-Leffler Function in the Kernel, Yokohama Math. J. 19 (1971), 7-15.
28. Purohit, S.D., Kalla, S. L. and Suthar, D. L., Fractional Integral Operators and the Multiindex Mittag-Leffler Function, Scientia Series A: Mathematical Science, Vol.21 (2011), 87-96.
29. Rainville, E.D., Special Functions, Chelsea Publ. Co. Bronx, New York (First Published by Macmillan, New York (1960)).
30. Rao, Snehal B., Jyotindra C. Prajapati, Amitkumar D. Patel, and Ajay K. Shukla. "Some properties of Wright-type generalized hypergeometric function via fractional calculus." *Advances in Difference Equations* 2014, no. 1 (2014): 119.
31. Ross, B., A Brief History and Exposition of the Fundamental Theory of Fractional Calculus: Fractional Calculus and Its Applications (B. Ross Ed.). Springer-Verlag, Berlin, Heidelberg and New York (1975).
32. Samko, S. G., Kilbas, A. and Marichev, O., Fractional Integrals and Derivatives: Theory and Application, Gardon and Breach, Yverdon, et al. (1993)
33. Saxena, R. K. and Nishimota, K., N- Fractional Calculus of Generalized Mittag-Leffler Functions, J. Fract. Calc, 37 (2010), 43-52.
34. Saxena, R. K. and Saigo, M., Certain Properties of the Fractional Calculus Operators Associated with Generalized M-L Function, Fact. Cal.Appl. Anal. Vol.8 No.2 (2005), 141-154.
35. Saxena, R.K., Fractional Integral Formula for the H- Function, Journal of Fractional Calculus Vol. 6, (1994), 65-75.
36. Saxena, R.K., Ram, J. and Vishnoi, M., Integral Representation of Generalized M-L Function of Fractional Calculus, Int. J. Math Sci. Appl. Vol.1 No.1 (2011), 1-13.
37. Sharma M. and Jain, R., Note on a Generalized M-series as a Special Function of Fractional Calculus, Fract. Calc. Appl. Anal.12 No.4 (2009), 449-452.
38. Sharma, K. and Dhakar, V.S., On Fractional Calculus and Certain Results Involving K_2-Function, GJSFR.Vo.11 Issue 5 Version 1.0 (2011), 17-21.
39. Sharma, K., A Solution of Generalized Fractional Volterra Type Equation Involving K_4-Function, General Mathematics Note, USA, (2011).

40. Sharma, K., An Introduction to the Generalized Fractional Integration, Bol. Soc. Paran Mat. Vol. 30.No.2 (2012), 1-5.
41. Sharma, K., Jain, R. and Sharma, M., Fractional q-Derivatives and Basic Hypergeometric Functions, NIARJS Vol.6 (2011), p37-38.
42. Sharma, K., On Application of Fractional Differintegration Operator to the K_4-Function, Bol. Soc. Paran Mat. Vol. 30.No.1 (2012), 91-97.
43. Sharma, K., On the Solution of Generalized Fractional Kinetic Equation. Involving the Functions for the Fractional Calculus GJSFR. Vol.11 Issue 6 Version 1.0 (2011).
44. Sharma, K., Some properties of a generalized Mittag-Leffler type functions, J. Indian Math. Soc. 82(1-2), (2015), 155-167.
45. Sharma, M., Fractional Integration and Fractional Differential of the M-series, Fract. Cal. Appl.Anal 11(2008), 187-191.
46. Shukla, A. K. and Prajapati, J. C., On a Recurrence Relation of Generalized Mittag-Leffler Function, Surveys in Mathematics and its Applications Vol.4 (2009), 133-138.
47. Shukla, A. K. and Prajapati, J. C., Some Remarks on Generalized Mittag-Leffler-Function, Proyecciones Vol.28 No.1 (2009), 27-34.
48. Shukla, A.K. and Prajapati, J.C., On a Generalization of Mittag-Leffler Function and its Properties, J. Reine. Angew. Math. Anal.Appl.336 (2007), 797-811.
49. Slater, L.J., Generalized Hypergeometric Functions, Cambridge University Press, Cambridge (1966).
50. Srivastava, H. M. and Tomovski, Z., Fractional Calculus with an Integral Operator Containing a Generalized M-L function in the Kernel Appl. Math.comput. 21 (2009), 198-210.
51. Srivastava, H.M. and Buschman, R.G., Convolution Integral Equations with Special Function Kernels, Halsted Press (Wiley Eastern Limited, New Delhi), John Wiley and Sons, New York, London, Sydney and Toronto (1977).
52. Srivastava, H.M. and Panda, R., Some Bilaterm Generating Function for a Class of Generalized Hypergeomertic Polynomials, J. Reine. Angew. Math. Anal.Appl.283/284 (1996) 265-274.
53. Srivastava, H.M., Gupta, K.C. and Goyal, S.P., The H-Function of One and Two Variables with Applications, South Asian Publishers (1982).

54. Srivastava, H.S., Certain Class of Eulerian Integrals of Multivariable Generalized Hyper geometric Function, J. Indian Acad. Math. Vol. 21 (1999) 255-270.
55. Watugala, G. K., Sumudu Transform: A New Integral Transform to Solve Differential Equations and Control Engineering Problems, Int. J. Math. Edu.Sci.Tech. 24 (1993), 35–43.